GROUP COHOMOLOGY

AKHIL MATHEW

1. COHOMOLOGY AND HOMOLOGY

Let $G$ be a group. We can form the group ring $\mathbb{Z}[G]$ over $G$; by definition it is the set of formal finite sums $\sum a_i g_i$, where $a_i \in \mathbb{Z}, g_i \in G$, and multiplication is defined in the obvious manner.

We shall call an abelian group $A$ a $G$-module if it is a left $\mathbb{Z}[G]$-module. This means, of course, that there exists a homomorphism $G \to \text{Aut}_G(A)$. We can also make $A$ into a right $\mathbb{Z}[G]$-module simply by writing $ag := g^{-1}a$ for $a \in A, g \in G$. This is important for tensor products. An example of a $G$-module is any abelian group with trivial action by $G$. For instance, we shall in the future denote by $\mathbb{Z}$ the integers with trivial $G$-action. Finally, if $A$ and $B$ are $G$-modules, then a $G$-homomorphism between them is a map $\phi : A \to B$ which is a $\mathbb{Z}[G]$ homomorphism. The set of $G$-homomorphisms between $A$ and $B$ is denoted by $\text{Hom}_G(A, B)$. It is a left exact functor of $A$ and $B$, covariant in $B$ and contravariant in $A$.

As usual its derived functors are denoted by $\text{Ext}^i$.

Let $A$ be a $G$-module. Then we define the cohomology groups as

$$H^i(G, A) := \text{Ext}^i_{\mathbb{Z}[G]}(\mathbb{Z}, A).$$

Then the $H^i(G, \cdot)$ are covariant functors from the category of $G$-modules to the category of abelian groups. Now we have clearly $H^0(G, A) = \text{Hom}_G(\mathbb{Z}, A)$, by basic properties of Ext over any ring. Also, a $\mathbb{Z}$-homomorphism $\mathbb{Z} \to A$ is determined by its value at 1; it is a $G$-homomorphism if and only if its image $a \in A$ is fixed by $G$, i.e. $ga = a$ for all $g \in G$. Denote the set of such $a$ by $A^G$. So we see that $\text{Hom}_G(\mathbb{Z}, A) = A^G$; in particular $A \to A^G$ is a left exact functor. Then another way of stating our definitions is that $H^i(G, \cdot)$ are the derived functors of the functor $A \to A^G$.

Go back to the initial definition of $H^i(G, \cdot)$ in terms of Ext. Let $\{P_i\}$ be a projective resolution of $\mathbb{Z}$, i.e. an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to \mathbb{Z},$$

where all the $P_i$ are projective (e.g. free) $G$-modules. Then we have a chain complex

$$0 \to \text{Hom}_G(P_0, A) \to \text{Hom}_G(P_1, A) \to \cdots$$

The homology groups of this complex are the derived functors $H^i(G, A)$.

The above characterization yields the following:

**Proposition 1.** Let $A$ be a co-induced $G$-module, i.e. $A = \text{Hom}_G(\mathbb{Z}[G], B)$ for a $\mathbb{Z}$-module $B$. (This can be made into a $G$-module by multiplication on the right.) Then we have $H^i(G, A) = 0$ if $i > 0$.

**Proof.** Consider the chain complex $\text{Hom}_G(P_i, A) = \text{Hom}_G(P_i, \text{Hom}_G(\mathbb{Z}[G], B)) = \text{Hom}_G(P_i, B)$, as is easily seen. Since the $P_i$ are projective, this sequence is exact except possibly at $i = 0$, and the proposition follows. □
Moreover, by basic properties of derived functors, we have the following result: If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a “short” exact sequence of $G$-modules, then there exists a “long” exact sequence

$$0 \rightarrow A^G \rightarrow B^G \rightarrow C^G \rightarrow H^1(G, A) \rightarrow H^1(G, B) \rightarrow H^1(G, C) \rightarrow H^2(G, A) \rightarrow H^2(G, B) \ldots .$$

The definition of homology is similar. Let $A$ be a $G$-module. We define

$$H_i(G, A) := \text{Tor}_{Z[G]}(Z, A).$$

Here we have used the making of $A$ into a right $Z[G]$-module, as above. Then by definition, the $H_i(G, \cdot)$ are the derived functors of the functor $A \mapsto Z \otimes_{Z[G]} A$, which is also $H_0(G, A)$, by elementary properties of Tor.

In the future, we suppress the $Z[G]$ in the tensor product $\otimes_{Z[G]}$, for convenience. Let us compute the functor $Z \otimes_{Z[G]} A$. We have an exact sequence of $G$-modules

$$0 \rightarrow I_G \rightarrow Z[G] \rightarrow Z \rightarrow 0,$$

where $I_G$ is the augmentation ideal, i.e. the $G$-submodule of $Z[G]$ generated by the $g - 1$, $g \in G$; and the homomorphism $Z[G] \rightarrow Z$ is the augmentation homomorphism that maps each $g$ to $1$. Since the tensor product is right exact, we have the following exact sequence:

$$I_G \otimes_{Z[G]} A \rightarrow Z[G] \otimes_{Z[G]} A \rightarrow Z \otimes_{Z[G]} A \rightarrow 0.$$

The second term is just $A$. Thus we see that $Z \otimes A = A/I_G A$. In other words, the $H_i(G, \cdot)$ are the derived functors of $A \mapsto A/I_G A$.

There are derived functor properties of the homology groups. As usual, if $\{P_i\}$ is a projective resolution of $Z$ as before, then the $H_i(G, A)$ are the homology groups of the chain complex (tensor products over $Z[G]$)

$$\cdots \rightarrow P_2 \otimes A \rightarrow P_1 \otimes A \rightarrow P_0 \otimes A \rightarrow 0.$$

This immediately yields the following analog of Proposition 1:

**Proposition 2.** Let $A$ be an induced $G$-module, i.e. $A = Z[G] \otimes_Z B$ for a $Z$-module $B$. (This can be made into a $G$-module by multiplication on the left.) Then we have $H_i(G, A) = 0$ if $i > 0$.

**Proof.** Indeed:

$$P_n \otimes_{Z[G]} A = P_n \otimes_{Z[G]} (Z[G] \otimes_Z B) = (P_n \otimes_{Z[G]} Z[G]) \otimes_Z B = P_n \otimes_Z B,$$

and the $P_n$ are projective. \(\square\)

We have a similar long exact sequence, which we leave the reader to write out.

2. Examples

We construct an explicit resolution of $Z$, which will give us a practical criterion for computing the cohomology and homology groups. Define $P_i$ to be the free abelian group generated by the symbols $(g_0, \ldots, g_i)$ for $g_j \in G, 0 \leq j \leq i$. Let $G$ act on $P_i$ in the obvious manner, i.e. $g(g_0, \ldots, g_i) := (g g_0, \ldots, g g_i)$. Then the $P_i$ are $G$-modules. We give the boundary homomorphisms. The homomorphism $P_0 \rightarrow Z$ maps $(g_0) \rightarrow 1$ for all $g_0 \in G$, and is extended by linearity. The homomorphism $d_i : P_i \rightarrow P_{i-1}$ for $i > 0$ is given by

$$d_i(g_0, \ldots, g_i) := \sum_{j=0}^{i} (-1)^j (g_0, \ldots, \hat{g}_j, \ldots, g_i),$$
where as usual the hat means a term is omitted. These are $G$-homomorphisms and form a chain complex (compute $d_i \circ d_{i-1}$ directly).

We must now check exactness. First, if $d_0(a) = 0$, then we can write $a$ as a sum of $(g_j) - (h_j)$, which is the boundary of $\sum (g_j, h_j)$. Next, fix $g \in G$. We define a chain homotopy $D_i : P_i \to P_{i+1}$ by $D_i(g_0, \ldots, g_i) := (g, g_0, \ldots, g_i)$. We see easily that $Dd + dD = 1$, by abuse of notation, whence the sequence is exact. In the future we will drop indexes like that.

Hence, for instance, a $n$-cocycle in some $G$-module $A$ is a $G$-homomorphism $f : P_n \to A$ such that $fd = 0$. $f$ is a $n$-coboundary if $f = gd$ for some $G$-homomorphism $P_{n-1} \to A$.

There is, however, a more convenient notation. $P_i$ is a free $G$-module on the set $[g_1, \ldots, g_i] := \{1, g_1, g_1g_2, g_1g_2g_3, \ldots, g_1 \ldots g_i \}$. So a $G$-homomorphism $f : P_i \to A$ is equivalent to a map assigning an element of $A$ to each $[g_1, \ldots, g_i]$, which extends to the free module in the usual manner. Note that

$$d[g_1, \ldots, g_i] = g_1[g_2, \ldots, g_i] + \sum_{j=1}^{i-1} (-1)^j [g_1, \ldots, g_jg_{j+1}, \ldots, g_i] + (-1)^i [g_1, \ldots, g_{i-1}].$$

Now denote the set of $[g_1, \ldots, g_i]$ by $S_i$. Hence a $n$-cocycle becomes a map $f : S_i \to A$ such that

$$g_1f([g_2, \ldots, g_{i+1}]) + \sum_{j=1}^{i} (-1)^j f([g_1, \ldots, g_jg_{j+1}, \ldots, g_i]) + (-1)^{i+1} f([g_1, \ldots, g_i]) = 0,$$

where a $n$-coboundary becomes a map $f$ such that there exists a map $h : S_{i-1} \to A$ with

$$f([g_1, \ldots, g_i]) = g_1h([g_2, \ldots, g_i]) + \sum_{j=1}^{i-1} (-1)^j h([g_1, \ldots, g_jg_{j+1}, \ldots, g_i]) + (-1)^i h([g_1, \ldots, g_{i-1}]).$$

Take the case $i = 1$. Then a 1-cocyle is a map $f : G \to A$ such that $f(st) = sf(t) + f(s)$, or a crossed homomorphism. A 1-coboundary is a map $f$ of the form $f(s) = sa - a$. This allows us to compute the cohomology groups in special cases.

We will use one more example, this time for homology:

**Proposition 3.** $H_1(G, \mathbb{Z}) = G_{ab}$, the abelianization of $G$.

**Proof.** We have an exact sequence

$$0 \to I_G \to \mathbb{Z}[G] \to \mathbb{Z} \to 0,$$

with the usual augmentation ideal and augmentation homomorphism. Taking the long exact sequence and noting that $\mathbb{Z}[G]$ is $\mathbb{Z}[G]$-free (hence projective), we see that $H_1(G, \mathbb{Z}) = H_0(G, I_G) = I_G/I_G^2.$ So we have to define a map $f : G_{ab} \to I_G/I_G^2$ which is an isomorphism. Let $f(s) := s - 1$; then $f(st) - f(s) - f(t) = st - 1 - s - 1 - (t - 1) = 0$ in $I_G^2$, so $f$ is a homomorphism. For the inverse, define $g : I_G/I_G^2 \to G_{ab}$ by $g(\sigma - 1) := \sigma$. Note that all elements in $I_G/I_G^2$ are of that form. As above this is a homomorphism (since $G_{ab}$ is abelian), and it is clearly the inverse of $f$. \hfill $\square$

3. Inflation, Restriction, Corestriction

Let $A$ be a $G$-module, and let $f : G' \to G$ be a homomorphism of groups. We can consider $A$ as a $G'$ module by setting $g'a := f(g')a$ for $g' \in G'$. We have obviously $A^G \subseteq A^{G'}$. Hence there is an inclusion homomorphism $H^0(G, A) \to H^0(G', A)$. Now the families of functors $\{H^i(G, \cdot)\}$ and $\{H^i(G', \cdot)\}$ are universal $\delta$-functors in the sense
of Grothendieck, (see [?]) since they are derived functors. Hence this morphism extends uniquely to morphisms

\[ H^i(G, A) \to H^i(G', A), \quad i \geq 0, \]

which are, as usual, functorial in \( A \), and which commute with the boundary homomorphisms for long exact sequences generated by short exact sequences \( 0 \to A \to B \to C \to 0 \).

If \( H \subset G \) is a subgroup, and \( A \) is a \( H \)-module then we get a map

\[ \text{Res} : H^i(G, A) \to H^i(H, A). \]

This is called restriction. We can of course define it directly on the cochains and coboundaries, by a pull-back. That this gives the same restriction map is immediate from the uniqueness.

We can go in the opposite direction. Let \( H \subset G \) be a normal subgroup, and let \( A \) be a \( G \)-module. Then \( A^H \) is a \( G/H \)-module in the obvious manner. Indeed, if \( a \in A \) is fixed by the actions of \( H \), and \( g \in G, h \in H \), we have that \( h(ga) = gh'a = ga \) for some \( h' \in H \), since \( H \) is normal. This shows that \( A^H \) is a \( G/H \)-module. The canonical homomorphism \( G \to G/H \) induces a homomorphism

\[ H^i(G/H, A^H) \to H^i(G, A^H), \]

which when composed with the canonical map \( H^i(G, A^H) \to H^i(G, A) \) induced by inclusion, yields the inflation map:

\[ H^i(G/H, A^H) \to H^i(G, A), \]

which is, of course, a morphism of \( \delta \)-functors as well.

Both restriction and inflation can be computed directly via cochains as well, in an obvious manner.

**Proposition 4** (Restriction-Inflation Exact Sequence). Let \( A, H, G \) be as above. Then the sequence with inflation and restriction

\[ 0 \to H^1(G/H, A^H) \to H^1(G, A) \to H^1(H, A) \]

is exact.

**Proof.** Direct verification on cochains and coboundaries.

1. We first check that the first map is injective. If \( f : G/H \to A^H \) is a cochain, then the pull-back \( f_G : G \to A \) represents \( \text{Inf}(f) \). If \( f_G(s) = sa - a \) for some \( a \in A \), i.e. \( f_G \) is a coboundary, then it follows immediately from the fact that \( f_G \) is constant on cosets of \( H \) that \( sa - a = tsa - a \) for \( t \in H \), or \( a \in A^H \). It thus follows that \( f \) itself is a coboundary (through \( a \)).

2. Now we check that the composite of the two maps is zero. So choose a crossed homomorphism \( f : G/H \to A^H \). Since \( f(11) = f(1) + f(1) = 2f(1) \), we see that \( f(1) = 0 \). Lifting to \( G \) and denoting the inflation by \( f_G \) we see that \( f_G(s) = 0 \) for \( s \in H \); restricting to \( H \) we get the zero cocyle.

3. Now all we need is to show that if \( h : G \to A \) is a crossed homomorphism and there exists \( a \in A \) such that \( h(s) = sa - a \) for \( s \in H \), then \( h \) is an inflation of some crossed homomorphism \( f : G/H \to A \). By subtracting the coboundary \( sa - a \) from \( h \), we assume \( h \) vanishes on \( H \). Then we have \( h(st) = h(s) + sh(t) \) for any \( s, t \in G \). Choose \( t \in H \) so that \( h(t) = 0 \); then \( h(st) = h(s) \). We see that \( h \) is constant on the \( H \)-cosets. Now take \( s \in H \), and we get \( h(t) = h(st) = 0 + sh(t) \).
Thus \( h(t) \) is fixed under \( H \), so the values of \( H \) come from \( A^H \). This is enough to conclude that \( h \) is an inflation and concludes the proof.

\[ \square \]

**Proposition 1.** Let \( H \subset G \) be normal. Let \( q > 1 \), and suppose that \( H^i(H, A) = 0 \) if \( 1 \leq i \leq i - 1 \). Then we have an exact sequence:

\[
0 \to H^0(G/H, A^H) \to H^q(G/H, A) \to H^q(H, A).
\]

**Proof.** This follows from a simple dimension-shifting argument. Embed \( A \) in a \( G \)-coinduced module \( B \); let \( C \) be the cokernel. Then \( B \) is also \( H \)-coinduced. Moreover, we have an exact sequence

\[
0 \to A \to B \to C \to 0,
\]

by looking at the \( H \)-cohomology. One uses the commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & H^{q-1}(G/H, C^H) & \longrightarrow & H^{q-1}(G, C) & \longrightarrow & H^{q-1}(H, C) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^{q-1}(G/H, A^H) & \longrightarrow & H^{q-1}(G, A) & \longrightarrow & H^{q-1}(A, C).
\end{array}
\]

The vertical boundary maps are isomorphisms, hence the proposition follows by induction.

\[ \square \]

There is a similar definition for **corestriction**. Let \( A \) be a \( G \)-module, \( H \subset G \) not necessarily normal. One begins with the obvious map

\[
A/I_H A \to A/I_G A,
\]

and one gets morphisms

\[
H_i(H, A) \to H_i(G, A),
\]

for all \( i \), which are natural in the usual senses.

We can also define corestriction on the **cohomology** groups. Let \( H \subset G \). Then \( A^G \subset A^H \). Let \( S \) be a set of representatives of left cosets of \( H \) in \( G \), so that \( SH = G \). Define \( f : A^H \to A^G \) by:

\[
f(a) := \sum_{s \in S} sa.
\]

Then \( f \) defines the **corestriction** on the zeroth cohomology group. The map is clearly independent of the choice of \( S \). It extends to the corestriction maps on all the groups. There is a similar procedure for the restriction on the homology groups.

**Proposition 2.** We have \( Cor \circ Res = (G : H) \) on all \( H^i(G, A) \) (multiplication by \( (G : H) \)).

**Proof.** Trivial for \( i = 0 \). Dimension-shifting for \( i > 0 \).

\[ \square \]

### 4. The Tate Groups

Let \( G \) be a **finite group**, and \( A \) a \( G \)-module. We define \( N = \sum_{g \in G} g \in \mathbb{Z}[G] \). Clearly we have \( NA \subset A^G \), and \( I_G A \subset A_N \). (We use the notation \( A_f \) for a homomorphism of \( A \) into some other group to denote \( ker f \).) We get a homomorphism \( N'_A : H_0(G, A) \to H^0(G, A) \). Using this, we can split the homology and cohomology groups together. Indeed, we set:

1. \( \check{H}^i(G, A) := H^i(G, A) \) if \( i > 0 \).
Theorem 1. Suppose \( 0 \to A \to B \to C \to 0 \) is a short exact sequence. Then there exists a long exact sequence

\[
\cdots \to \hat{H}^i(G, A) \to \hat{H}^i(G, B) \to \hat{H}^i(G, C) \to \hat{H}^{i+1}(G, A) \to \hat{H}^{i+1}(G, B) \to \cdots.
\]

So the Tate groups form a \( \delta \)-functor of covariant functors. We show that it is universal. The next proposition shows this:

Proposition 3. Let \( A \) be an induced module, i.e. \( A = \mathbb{Z}[G] \otimes_{\mathbb{Z}} B \) for an abelian group \( B \). Then \( \hat{H}^i(G, A) = 0 \) for all \( i \).

In other words, \( A \) is cohomologically trivial.

Proof. Since \( G \) is finite, we see that \( A = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], B) \). Thus \( A \) is both induced and co-induced; hence the regular cohomology and homology groups for \( i > 0 \) vanish. Hence the only Tate groups we need consider are \( \hat{H}^0(G, A) \) and \( \hat{H}^{i-1}(G, A) \). These can be checked directly easily.

Restriction and corestriction extend to all the Tate groups, as is easy to see. Straightforward computations direct from the definitions show that they are natural transformations of \( \delta \)-functors. (The only place where this needs to be checked is at \( i = -1, 0 \) where the cohomology and homology groups are glued together. This is straightforward.)

We have \( \text{Cor} \circ \text{Res} = (G : H) \) as before, by dimension-shifting or uniqueness.

Proposition 4. Let \( n := (G : 1) \). Then the groups \( \hat{H}^i(G, A) \) have exponent dividing \( n \).

Proof. Immediate from the above discussion, taking \( H := 1 \).

Proposition 5. If \( A \) is finitely generated, then the Tate groups are finite.
Proof. Looking at the cochains (or chains) we see that the Tate groups are finitely generated. By the structure theorem for finitely generated abelian groups, our assertion follows at once from the previous proposition.

\[ \hat{H}^p(G, A) \times \hat{H}^q(G, B) \to \hat{H}^{p+q}(G, A \otimes B) \]

for all \( p, q \), which are functorial in \( A \) and \( B \). They satisfy the following properties:

1. The map \( \hat{H}^0(G, A) \times \hat{H}^0(G, B) \to \hat{H}^0(G, A \otimes B) \) comes from the canonical map \( A^G \times B^G \to (A \otimes B)^G \).

2. The map commutes with boundary homomorphisms in the following sense. Suppose \( 0 \to A' \to A \to A'' \to 0 \) is an exact sequence, which remains exact when tensored with the \( G \)-module \( B \). Then if \( a'' \in \hat{H}^p(G, A'') \), \( b \in \hat{H}^q(G, B) \), we have:

\[ (\delta a'').b = \delta(a''.b). \]

3. Conversely if \( 0 \to B' \to B \to B'' \to 0 \) is exact, and the sequence remains exact when tensored with \( A \), we then have:

\[ a.(\delta b'') = (-1)^p \delta(a.b''). \]

Proof. Cf. [?].

6. THE HERBRAND QUOTIENT

Let \( G \) be a cyclic group of order \( n \), generated by \( s \in G \). Then we define an exact complex \( C \):

\[
\cdots \xrightarrow{s-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{s-1} \mathbb{Z}[G] \xrightarrow{N} \mathbb{Z}[G] \xrightarrow{s-1} \cdots
\]

where \( N \) is the norm homomorphism. Define the functor \( F \) by \( F(A) := H(A \otimes C) \), i.e. the homology of \( A \otimes C \). This is immediately seen to be a \( \delta \)-functor of \( A \) since the modules \( \mathbb{Z}[G] \) are free, hence flat. The \( \delta \)-functor also coincides with the Tate groups at 0, hence it consists of the Tate groups since both \( \delta \)-functors are universal.

In particular, the Tate groups are periodic with period (dividing) two.

Define the Herbrand quotient as \( h(A) := \frac{\text{card}(\hat{H}^1(G,A))}{\text{card}(\hat{H}^0(G,A))} \). It is defined if both the numerator and the denominator are finite groups.

We can view the Herbrand quotient as an Euler-Poincare characteristic. Indeed, if \( 0 \to A \to B \to C \to 0 \) is a short exact sequence, then the usual long exact sequence becomes an exact hexagon, which I don’t know how to typeset. But from the exact hexagon it follows at once that the Herbrand quotient is multiplicative.

Note that \( h(A) = 1 \) if \( A \) is finite. Indeed, we have an exact sequence

\[ 0 \to A^G \to A \to A/I_GA \to 0, \]

where the second map is multiplication by \( s - 1 \) for \( s \) a generator of \( G \). We see that \( A^G \) and \( A_G \) have the same cardinality if \( A \) is finite.
7. GALOIS COHOMOLOGY

Let $L/K$ be a finite Galois extension with group $G$. Then $L$ is a $G$-module in the obvious manner. Note that $K = L^G$ by Galois theory.

**Theorem 3** (Normal Basis Theorem). The additive group $L$ is $G$-free. In other words, there exists a basis of $L/K$ given by $x_\sigma$, $\sigma \in G$, such that $\tau x_\sigma = x_{\tau\sigma}$ for $\tau \in G$.

**Proof.** See [?].

By the previous section, we get:

**Theorem 4.** $L$ is cohomologically trivial over $G$.

We next consider the multiplicative group $L^\times$. It is also a $G$-module. We denote $s(a)$ by $a^s$ for $s \in G$, for convenience.

**Theorem 5** (Hilbert 90). $H^1(G, L^\times) = 0$.

**Proof.** Let $f : G \to L^\times$ be a cocyle, i.e. $f(ts) = f(t)f(s)^t$ for all $s, t \in G$. Consider the sum

$$\sum_s f(s)s \in L[G],$$

which is obviously a $K$-endomorphism of $L$. It cannot be identically zero, since distinct $K$-automorphisms of $L$ are linearly independent by a theorem of Dedekind. (See [?].) So pick $a \in L^\times$ with:

$$b := \sum_s f(s)a^s \neq 0.$$

Now consider $b^t$:

$$b^t = \sum_s f(s)^t a^{ts} = \sum_s f(t)^{-1} f(ts)a^{ts} = f(t)^{-1}b,$$

or in other words:

$$f(t) = \frac{b}{b^t}.$$

Hence $f$ is a coboundary as well. □

8. COHOMOLOGICAL TRIVIALITY

Let $G$ be a finite group. We say that a $G$-module $A$ is cohomologically trivial if all its Tate groups vanish. An induced module is cohomologically trivial.

We now consider $p$-groups.

**Theorem 6.** Let $G$ be a $p$-group, $A$ a $G$-module annihilated by $p$. If we have $\hat{H}^i(G, A) = 0$ for some $i$, then $A$ is cohomologically trivial. Indeed $A$ is free over $F_p[G]$.

We need a lemma:

**Lemma 1** (Noncommutative Nakayama). Suppose $H_0(G, A) = 0$. Then $A = 0$.

**Proof.** We have $\hat{H}^{-1}(G, A) = H_0(G, A) = 0$, i.e. $A/I_G A = 0$. Consider now $A' := \text{Hom}_G(A, F_p)$. We then have $A'^G = 0$ since $A = I_G A$ and a $G$-homomorphism into $F_p$ must annihilate $I_G A$, assuming of course that $G$ acts trivially on $F_p$. Now $A'$ is a vector space over $F_p$ too. We show that $A' = 0$, hence $A = 0$. Indeed, consider a finitely generated $G$-submodule of $A'$, say $B'$. If $B' = 0$, this will establish our contention. Suppose $B' \neq 0$. But $B'$ is of order a power of $p$, and it is acted on by a group of order a power of $p$. Hence the number of fixed points is a power of $p$ and thus consists not only of the origin, so $B'^G \neq 0$, contradiction. □
As a simple corollary, if \( A = I_GA \), then \( A = 0 \). Alternatively, if \( A = I_GA + B \), then \( A = B \). This is strikingly reminiscent of Nakayama’s lemma.

**Proof.** Start with \( i = -2 \). We are given that \( H_1(G, A) = 0 \). We will show that \( A \) is free over \( F_p[G] \). Start by lifting a \( F_p \)-basis of \( A/I_GA \) to \( A \), say \( L = \{ u_1, \ldots, u_m \} \). Then \( L \) generates \( A \) by the above remarks. Let \( B \) be the free \( F_p[G] \)-module on \( L \). We have a surjective \( G \)-homomorphism \( B \to A \). Let \( B' \) be the kernel. Then we have an exact sequence

\[
0 \to B' \to B \to A \to 0,
\]

Now by “dimension-shifting”, we get \( H_0(G, B') = H_1(G, A) = 0 \). Indeed, we have an exact sequence

\[
0 = H_1(G, A) \to H_0(G, B') \to H_0(G, B) \to H_0(G, A).
\]

But \( B/I_GB = A/I_GA \), so the last two are isomorphic. Thus \( H_0(G, B') = 0 \). Hence \( B' = 0 \), and \( B \to A \) is an isomorphism. Hence \( A \) is free over \( F_p[G] \), proving the theorem in this case.

Now suppose \( i \neq 2 \). Use a repeated dimension-shifting argument to get a module \( B \) whose Tate groups \( \hat{H}^i(H, B) \) for any \( H \subset G \) are a translation of those of \( A \) (i.e. \( \hat{H}^i(H, A) \)), such that the zero module for \( A \) falls at \(-2\) for \( B \). Then it follows by the above argument that \( B \) is \( F_p[G] \)-free, hence cohomologically trivial. Since \( A \)'s Tate groups (for all subgroups \( H \)) are just a translation, \( A \) is also cohomologically trivial. \( \square \)

We now consider the opposite scenario.

**Theorem 7.** Let \( A \) be a \( G \)-module without \( p \)-torsion, \( G \) a \( p \)-group. Then \( A \) is cohomologically trivial if and only if the Tate groups \( \hat{H}^i(G, A) \) vanish for consecutive indices.

**Proof.** We have an exact sequence

\[
0 \to A \xrightarrow{p} A \to A/pA.
\]

Now \( A/pA \) has only \( p \)-torsion. If for two consecutive indices the Tate groups of \( A \), i.e. \( \hat{H}^i(G, A) \) vanish, then a Tate group \( \hat{H}^i(G, A) \) of \( A/pA \) also vanishes by the exact sequence of Tate groups. Hence \( A/pA \) is cohomologically trivial, so all its Tate groups \( \hat{H}^i(H, A/pA) = 0 \) for \( h \subset G \) and \( i \in \mathbb{Z} \). This implies again from the exact Tate group sequence that the maps

\[
\hat{H}^i(H, A) \xrightarrow{p} \hat{H}^i(H, A)
\]

are isomorphisms. But \( \hat{H}^i(H, A) \) has only \( p \)-torsion. (Composing \( \text{Cor} \) and \( \text{Res} \) from the unit group gives a power of \( p \).) Hence all the Tate groups of \( A \) vanish, proving the theorem. \( \square \)

**Theorem 8.** Let \( A \) be a \( G \)-module. For each prime \( p \), let \( G_p \) be a \( p \)-Sylow subgroup. Suppose that for each \( p \), we have \( \hat{H}^n_p(G_p, A) = \hat{H}^{n+p}(G_p, A) = 0 \). Then the Tate groups \( \hat{H}^i(G, A) = 0 \).

**Proof.** We see immediately from the previous results that \( \hat{H}^i(G_p, A) = 0 \) for all \( i \).

Now the composition \( \text{Cor} \circ \text{Res} \) with respect to a subgroup \( G_p \) is just multiplication by a power of \( p \), and it is zero. Hence the ideal

\[
I = \{ a : a \text{ annihilates the Tate groups } \hat{H}^i(G, A) \text{ for all } H, i \}
\]

contains a power of each prime, hence is the unit ideal, whence the theorem. In reality \( A \) is cohomologically trivial, but we won’t use this. \( \square \)
9. Tate’s Theorem

**Theorem 9.** Let \( f : A \to B \) be a \( G \)-homomorphism. Suppose that the induced homomorphisms (still denoted by \( f \)) on the Tate groups are as follows: for each \( p \), there is a \( p \)-Sylow subgroup \( G_p \) and an integer \( n_p \) such that \( f \) is surjective \( \hat{H}^{n_p}(G,A) \to \hat{H}^{n_p}(G,A) \), bijective for \( n_p + 1 \), and injective for \( n_p + 2 \). Then \( f \) induces isomorphisms on the cohomology groups.

**Proof.** Let \( A' := \mathbb{Z}[G] \otimes \mathbb{Z} A \) be the standard induced module containing \( A \). Let \( \bar{A} : +A' \oplus B \) (direct sum). We have an injective map \( A \to \bar{A} \) given by the canonical map direct summed with \( f \). Let \( C \) be the cokernel. There is a short exact sequence

\[
0 \to A \to \bar{A} \to C \to 0,
\]

which induces a long exact sequence

\[
\cdots \to \hat{H}^i(H,A) \to \hat{H}^i(H,\bar{A}) \to \hat{H}^i(H,C) \to \hat{H}^{i+1}(H,A) \to \cdots,
\]

where \( H \subset G \) is a subgroup. Now the cohomology of \( \bar{A} \) is the same as that of \( B \). Take \( H = G_p \). Writing out the exact sequence and using the assumptions shows by earlier results that \( C \) is cohomologically trivial, whence the result. \( \square \)

Let \( G \) be a finite group. Let \( A \) be a \( G \)-module. We have a bilinear map \( A \times \mathbb{Z} \to A \), which is trivially a \( G \)-homomorphism (with \( G \) acting trivially on \( \mathbb{Z} \)). Now let \( w \in \hat{H}^A(G,q) \). Cupping with \( w \) gives a morphism

\[
\hat{H}^i(G,\mathbb{Z}) \to \hat{H}^{i+q}(G,A \otimes \mathbb{Z} \mathbb{Z}), \quad \text{all } i.
\]

In other words, we get a map:

\[
\hat{H}^i(G,\mathbb{Z}) \to \hat{H}^{i+q}(G,A).
\]

If \( q = 0 \), then this map just comes from \( \mathbb{Z} \to A \otimes \mathbb{Z} \), which follows for \( i = 0 \) by the definitions of cup-products, and then by dimension-shifting.

**Theorem 10.** Fix \( w \) as above. Suppose that for each \( p \), the cup-product of \( \text{Res}w \) is injective on \( \hat{H}^{n_p}(G_p,A) \), bijective on the next group, and surjective on the one after that. (Same notation as before.) Then the cup-product induces isomorphisms from \( \hat{H}^i(G,\mathbb{Z}) \to \hat{H}^{i+q}(G,\mathbb{Z}) \) for all \( i \).

For \( q = 0 \) this is the previous result. This can be seen by a dimension-shifting argument, and is of course plausible. See [?]. Now find an induced module \( A' \) and an exact sequence

\[
0 \to A \to A' \to A'' \to 0.
\]

We then have boundary isomorphic maps \( \hat{H}^i(G,A'') \to \hat{H}^{i+1}(G,A) \). Fix \( w \) now as before. Then \( \delta(w.a'') = \delta w.a'' \). This is better explained on paper.